

A covariant gauge-invariant three-dimensional description of relativistic bound-states *

D. R. Phillips and S. J. Wallace [†]

*Department of Physics and Center for Theoretical Physics,
University of Maryland, College Park, MD, 20742-4111
(February 9, 2008)*

Abstract

A formalism is presented which allows covariant three-dimensional bound-state equations to be derived systematically from four-dimensional ones without the use of delta-functions. The amplitude for the interaction of a bound state described by these equations with an electromagnetic probe is constructed. This amplitude is shown to be gauge invariant if the formalism is truncated at the same coupling-constant order in both the interaction kernel of the integral equation and the electromagnetic current operator.

I. INTRODUCTION

The work of Fleischer and Tjon [1–3] over twenty years ago provided a foundation for use of the Bethe-Salpeter (BS) formalism for nucleon-nucleon (NN) scattering. Calculations of electron-deuteron scattering using these BS wave functions were then performed by Zuilhof and Tjon [4,5]. Later work of van Faassen and Tjon [6,7] developed a one-boson-exchange model for the coupled $NN - N\Delta$ system based on the BS formalism that provides a reasonable description of the experimental NN phase shift data up to about 800 MeV.

Given this progress in four-dimensional calculations, one might ask whether three-dimensional calculations of the two-nucleon system are still useful. Reductions of the BS equation to a three-dimensional equation are motivated by several considerations. Firstly, three-dimensional calculations are simpler and hence allow the application of models for nucleon-nucleon interactions in three-body problems, e.g. the three-nucleon system and pion-deuteron scattering. In contrast, the only work using four-dimensional equations in three-hadron systems is an analysis for the three-nucleon system, by Rupp and Tjon [8,9], which used separable interactions and a nonrelativistic treatment of spin degrees of freedom. An analysis of the three-nucleon system with the same ingredients as the BS analyses of the two-nucleon problem has not been realized due to the technical complexities involved. Secondly, the meson-nucleon dynamics in NN interaction models involves ‘effective’ degrees of

*Dedicated to Professor J. A. Tjon on the occasion of his 60th birthday.

[†]Email: phillips@quark.umd.edu, wallace@quark.umd.edu.

freedom that are designed for use in low-order truncations of the full BS kernel. If the degrees of freedom are truly “effective” then one cannot really regard the meson-nucleon coupling constants as fundamental (with the exception of the πNN coupling constant). Given this point of view it is equally valid to construct an effective nucleon-nucleon potential in three dimensions, with the understanding that the coupling constants used will be different to those in the four-dimensional approach. Nevertheless, almost equivalent results for physical observables can be obtained in the three-dimensional and four-dimensional formalisms by making similar truncations of the scattering kernel and then adjusting the meson-nucleon coupling constants in both formalisms so that the empirical NN phase shifts are well described. Thirdly, while it might appear that relativistic effects cannot be correctly treated in a three-dimensional approach, in fact all relativistic effects can, in principle, be studied in the three-dimensional framework.

If relativistic effects are to be studied in a three-dimensional formalism some systematic reduction from the four-dimensional Bethe-Salpeter equation to a three-dimensional integral equation must be employed. Furthermore, if electromagnetic observables are of interest, this reduction should be implemented so that gauge invariance is realized in the three-dimensional approach. In this paper we present a three-dimensional formalism that is covariant and, after consistent truncation of the current operator and interaction kernel, is gauge invariant at any desired order.

Consider the Bethe-Salpeter equation (BSE) for the four-dimensional, covariant, two-to-two amplitude T [10–14],

$$T = K + KG_0T, \quad (1.1)$$

where G_0 is the free two-particle propagator, which in our convention is

$$G_0(p'_1, p'_2; p_1, p_2) = i(2\pi)^8 \delta^{(4)}(p'_1 - p_1) \delta^{(4)}(p'_2 - p_2) G_0(p; P), \quad (1.2)$$

$$G_0(p; P) = d_1(p_1) d_2(p_2), \quad (1.3)$$

with

$$d_i(p_i) = \frac{\Lambda_i^+(\mathbf{p}_i)}{p_i^0 - \epsilon_i(\mathbf{p}_i) + i\eta} - \frac{\Lambda_i^-(\mathbf{p}_i)}{p_i^0 + \epsilon_i(\mathbf{p}_i) - i\eta}, \quad (1.4)$$

where

$$\epsilon_i(\mathbf{p}_i) = \sqrt{\mathbf{p}_i^2 + m_i^2}, \quad (1.5)$$

$$\Lambda_i^\pm(\mathbf{p}_i) = \begin{cases} \frac{1}{2\epsilon_i(\mathbf{p}_i)}, & \text{for spin-zero particles,} \\ \frac{\pm \epsilon_i(\mathbf{p}_i) \gamma^0 - \gamma_i \cdot \mathbf{p}_i + m_i}{2\epsilon_i(\mathbf{p}_i)}, & \text{for spin-half particles.} \end{cases} \quad (1.6)$$

Here we have used the standard center-of-mass and relative four-momenta:

$$p_1 = \nu_1 P + p; \quad p_2 = \nu_2 P - p, \quad (1.7)$$

with $\nu_1 + \nu_2 = 1$. Throughout this paper, when functions are written with P and p as their arguments they do not contain four-momentum conserving delta functions. In comparison, when written with p_1 and p_2 as their arguments they do contain such delta functions. In

Eq. (1.4) η is a positive infinitesimal. In Eq. (1.1), K is the Bethe-Salpeter kernel. In principle, K should include all two-particle irreducible two-to-two Feynman graphs. Solution of (1.1) with the full two-particle irreducible kernel is impractical and usually the kernel is truncated to lowest order in the coupling constant, using the ladder approximation, $K = V$.

A simple way to obtain an approximate three-dimensional equation from Eq. (1.1) is to ignore the relative-energy dependence of the interaction V in the loop integral of the ladder BSE. This leads directly to the Salpeter equation [15], which, if only positive-energy states are considered, is equivalent to the equation of Blankenbecler-Sugar [16] and Logunov-Tavkhelidze [17]

$$T = V_{\text{inst}} + V_{\text{inst}}\langle G_0 \rangle T, \quad (1.8)$$

with the three-dimensional propagator

$$\langle G_0 \rangle = \int \frac{dp_0}{2\pi} G_0(p; P). \quad (1.9)$$

Tjon and collaborators have analyzed various hadronic systems, including elastic and inelastic electron-deuteron scattering [18–20], proton-proton bremsstrahlung [21], and the quark structure of hadrons [22–24], using a version of this equation with a modified three-dimensional propagator. These calculations have provided much insight into the role of relativity in few-body systems.

In order to avoid the instant approximation in the three-dimensional formalism, it is necessary to correct for the difference between the full four-dimensional interaction V (or K) and the instant version thereof. Moreover, Eq. (1.8) is usually obtained in the center-of-mass frame of the two-body system and the manner in which the amplitude is to be boosted to a frame other than the c.m. frame must be defined. Much work to address these issues has been based upon the quasi-potential formalism in which a delta function constrains the relative energy of the interacting particles. Here, we discuss a different reduction to three dimensions that does not possess the unphysical singularities which arise in quasipotential reductions.

A systematic formalism is found by splitting K into two pieces, one of which, K_1 , does not depend on the zeroth component of relative four-momentum, and the other of which, $K - K_1$, does. Straightforward manipulations then lead to the following set of equations (see [25] for details):

$$\Gamma = [1 - (K - K_1)G_0]^{-1} \Gamma_1; \quad (1.10)$$

$$\Gamma_1 = K_1 \langle \mathcal{G} \rangle \Gamma_1; \quad (1.11)$$

$$\mathcal{G} = G_0 + G_0(K - K_1)\mathcal{G}; \quad (1.12)$$

$$\langle \mathcal{G} \rangle \equiv \int \frac{dp'_0 dp_0}{(2\pi)^2} \mathcal{G}(p'_\mu, p_\mu; P). \quad (1.13)$$

There is now a general way to account for all the relative-energy integrations that reside in $\langle \mathcal{G} \rangle$. Demanding that K_1 be chosen such that $K_1 \langle \mathcal{G} \rangle = K_1 \langle G_0 \rangle$ produces

$$\Gamma_1 = K_1 \langle G_0 \rangle \Gamma_1; \quad (1.14)$$

$$K_1 \equiv \langle G_0 \rangle^{-1} \langle G_0 K \mathcal{G} \rangle \langle G_0 \rangle^{-1}. \quad (1.15)$$

This choice of K_1 is closely related to that used in the work of Klein [26–29] on three-dimensional reductions of four-dimensional equations, and to standard time-ordered perturbation theory.

This formalism was developed in a non-covariant way in Ref. [25]. In this paper we first show, in Section II, how a simple modification of this argument allows one to obtain a covariant formalism. (Note that in contrast to Ref. [25] we do not discuss the approximate inclusion of crossed-ladder terms in the kernel of the integral equation here, although it is straightforward to apply the discussion of Section II to that aspect of Ref. [25] also.) In Section III we take up the main issue of the paper: construction of a gauge-invariant amplitude in this formalism. We review the analysis of gauge invariance for the Green’s function of the Bethe-Salpeter equation:

$$G = G_0 + G_0 K G. \quad (1.16)$$

Gauging such four-dimensional Green’s functions is relatively straightforward. The critical issue is whether gauge invariance can be maintained within a consistent reduction scheme to three dimensions. We show how this is done in our approach to three-dimensional reduction, thereby demonstrating that gauge invariance can be maintained order-by-order by consistently truncating the kernel of the bound-state equation and the bound-state current matrix element.

II. COVARIANT BOUND-STATE EQUATIONS IN THREE DIMENSIONS

In this section we generalize the procedure presented in Ref. [25] to yield a covariant procedure for obtaining three-dimensional bound-state equations. While the technique is essentially that discussed in Ref. [25] we display it here for completeness and clarity.

Consider the Bethe-Salpeter equation for the bound-state vertex function, which we write as:

$$\Gamma(p; P) = \int \frac{d^4 p'}{(2\pi)^4} K(p, p'; P) G_0(p'; P) \Gamma(p'; P). \quad (2.1)$$

As discussed in the Introduction we now split the kernel K via

$$K(p, p'; P) = K_1(p, p'; P) + K_2(p, p'; P). \quad (2.2)$$

However, in contrast to the case discussed in the Introduction, here we make this decomposition in a covariant way. To do this we must write the relative four-momenta p and p' as

$$p = p_{\parallel D} \hat{D} + p_{\perp D}; \quad p' = p'_{\parallel D} \hat{D} + p'_{\perp D}. \quad (2.3)$$

Here, for any four-vector k , $k_{\parallel} = k \cdot \hat{D}$, with \hat{D} a unit four-vector in the direction of D . The piece of the kernel K_1 is then chosen so that it does not depend on $p_{\perp D}$ or $p'_{\perp D}$.

The choice of vector D is completely arbitrary and nothing depends upon it if the kernel used in the three-dimensional equation is treated exactly. Ultimately, of course, the kernel

must be truncated, and this introduces some dependence on the vector D into the three-dimensional formalism. This will be discussed further at the end of this section.

If the non-covariant vector

$$\hat{D} = (1, 0, 0, 0), \quad (2.4)$$

is chosen, then the reduction to three dimensions is exactly that developed in Ref. [25]. In order to obtain a covariant formalism for the bound-state problem, $D = P$ must be chosen, because P is the only available four-vector of the problem that is conserved. With this choice, Eq. (2.2) becomes a manifestly covariant decomposition of the kernel K , since then it is expressed solely in terms of Lorentz-covariant objects. Obviously, this choice coincides with Eq. (2.4) in the center-of-mass frame of the two-body system. But, in all other frames, this decomposition differs from that of Ref. [25], since it boosts from the c.m. frame in a covariant way. We keep D distinct from P in what follows in order that other choices of D may be considered for current matrix elements.

If Γ_1 is now defined by

$$\Gamma_1(p; P) = \int \frac{d^4 p'}{(2\pi)^4} K_1(p, p'; P) G_0(p'; P) \Gamma(p'; P), \quad (2.5)$$

it follows that, since K_1 does not depend on $p_{\parallel D}$, neither does Γ_1 . It is then a simple matter to show that Γ_1 obeys the equation:

$$\Gamma_1(p_{\perp D}; P) = \int \frac{d^4 p' d^4 p''}{(2\pi)^8} K_1(p_{\perp D}, p'_{\perp D}; P) \mathcal{G}(p', p''; P) \Gamma_1(p''_{\perp D}; P), \quad (2.6)$$

where the four-dimensional Green's function \mathcal{G} is defined by the integral equation

$$\mathcal{G}(p', p''; P) = G_0(p'; P) \delta^{(4)}(p' - p'') + \int \frac{d^4 p'''}{(2\pi)^4} G_0(p'; P) K_2(p', p'''; P) \mathcal{G}(p''', p''; P). \quad (2.7)$$

Since \mathcal{G} is the only four-dimensional object appearing on the right-hand side of Eq. (2.6), the equation reduces to one involving only three-dimensional integrals:

$$\Gamma_1(p_{\perp D}; P) = \int \frac{d^3 p'_{\perp D} d^3 p''_{\perp D}}{(2\pi)^6} K_1(p_{\perp D}, p'_{\perp D}; P) \langle \mathcal{G} \rangle(p'_{\perp D}, p''_{\perp D}; P) \Gamma_1(p''_{\perp D}; P). \quad (2.8)$$

Here, and in the rest of the paper, the notation $\langle \mathcal{G} \rangle$ denotes integration of the Green's function \mathcal{G} over four-momentum components parallel to D , in order to yield a three-dimensional quantity:

$$\langle \mathcal{G} \rangle(p'_{\perp D}, p''_{\perp D}; P) \equiv \int \frac{dp'_{\parallel D} dp''_{\parallel D}}{(2\pi)^2} \mathcal{G}(p', p''; P). \quad (2.9)$$

In order to obtain Γ from Γ_1 the equation

$$\Gamma(p; P) = \Gamma_1(p_{\perp D}; P) + \int \frac{d^4 p'}{(2\pi)^4} K_2(p, p'; P) G_0(p'; P) \Gamma(p'; P). \quad (2.10)$$

must be used. The Γ obtained from solving Eq. (2.8) and using Eq. (2.10) is identically equal to that found by solving the original Bethe-Salpeter equation (2.1). The difficulty is that the integral equations (2.7) and (2.10) are themselves four-dimensional. So, at this stage we have not simplified the problem at all.

We now consider whether there is a choice for K_1 which simplifies the form of \mathcal{G} . In particular, we seek a K_1 such that the propagator $\langle \mathcal{G} \rangle$ in Eq. (2.8) may be replaced by the free propagator $\langle G_0 \rangle$. In other words, the contribution of the last term of Eq. (2.7) must vanish once the integrals over $p'_{\parallel D}$ and $p''_{\parallel D}$ are performed. Writing $K_2 = K - K_1$ then leads to the defining condition:

$$\langle G_0 \rangle(p_{\perp D}; P) K_1(p_{\perp D}, p'_{\perp D}; P) \langle G_0 \rangle(p'_{\perp D}; P) = \left\langle \int \frac{d^4 p''}{(2\pi)^4} G_0(p; P) K(p, p''; P) \mathcal{G}(p'', p'; P) \right\rangle \quad (2.11)$$

with \mathcal{G} still defined in terms of K_1 by Eq. (2.7). With this choice of K_1 Eq. (2.8) involves a simple propagator:

$$\Gamma_1(p_{\perp D}; P) = \int \frac{d^3 p'_{\perp D}}{(2\pi)^3} K_1(p_{\perp D}, p'_{\perp D}; P) \langle G_0 \rangle(p'_{\perp D}; P) \Gamma_1(p'_{\perp D}; P), \quad (2.12)$$

and all the complexity is transferred to the interaction kernel K_1 . We write the equation (2.12) formally as:

$$\Gamma_1 = K_1 \langle G_0 \rangle \Gamma_1. \quad (2.13)$$

In order to calculate the kernel K_1 defined by Eq. (2.11), a perturbative expansion of the four-dimensional kernel, K , must be made:

$$K = \sum_{i=1}^{\infty} K^{(2i)}. \quad (2.14)$$

Then, to second order in the coupling constant we see that K_1 is given by

$$K_1^{(2)} = \langle G_0 \rangle^{-1} \langle G_0 K^{(2)} G_0 \rangle \langle G_0 \rangle^{-1}, \quad (2.15)$$

while at fourth order in the coupling constant we have

$$K_1^{(4)} = \langle G_0 \rangle^{-1} \left(\langle G_0 K^{(4)} G_0 \rangle + \langle G_0 K^{(2)} G_0 K^{(2)} G_0 \rangle \right) \langle G_0 \rangle^{-1} - K_1^{(2)} \langle G_0 \rangle K_1^{(2)}. \quad (2.16)$$

Here the propagator $\langle G_0 \rangle$ is

$$\langle G_0 \rangle(p_{\perp D}; P^2) = \frac{\Lambda_1^+ \Lambda_2^+}{P_{\parallel D} + i\eta - \epsilon_1 - \epsilon_2} - \frac{\Lambda_1^- \Lambda_2^-}{P_{\parallel D} - i\eta + \epsilon_1 + \epsilon_2}, \quad (2.17)$$

where $\Lambda_i^{\pm} = \Lambda_i^{\pm}(p_{i\perp D})$. These operators are defined as in Eq. (1.6), but with ϵ_i now given by $\epsilon_i(p_{i\perp D}) \equiv \sqrt{m_i^2 - p_{i\perp D}^2}$. In other words, ϵ 's and Λ 's are now defined as functions of $p_{i\perp D}$, not \mathbf{p}_i . These definitions then only coincide with the usual definitions of these quantities in the two-body center of mass frame.

The form (2.17) reveals an inconsistency in Eqs. (2.15) and (2.16). In a spin-half theory the inverse of $\langle G_0 \rangle$ does not exist, since the propagator has no components in sectors where one particle is in a positive-energy state and the other is in a negative-energy state. However, if we only take and use matrix elements of equations such as (2.15) and (2.16) in $++$ and $--$ states the results given above are entirely correct. This caveat must be borne in mind when examining the equations presented below. The difficulty springs from demanding that $\langle \mathcal{G} \rangle = \langle G_0 \rangle$ when the entity on the right-hand side of this equation does not allow propagation in $+-$ or $-+$ states. This problem can be completely avoided by use of the modified free propagator of Refs. [18–20,22,23,25,30–32], for which $\langle G_0 \rangle^{-1}$ is uniquely defined in all ρ -spin sectors. The results of this paper may be extended to that case. However, in order to focus our discussion on covariance and gauge invariance, we have chosen not to discuss this extension of the formalism here.

If we consider a spin-zero or spin-half field theory with a Yukawa interaction, then the interaction $K_1^{(2)}$ takes the form displayed in Ref. [25], up to trivial kinematic replacements:

$$\begin{aligned}
\langle G_0 \rangle(p'_{\perp D}; P) K_1^{(2)}(p'_{\perp D}, p_{\perp D}; P) \langle G_0 \rangle(p_{\perp D}; P) = \\
\frac{g_1 g_2 \mathcal{M}}{2\omega} \left[\left(\frac{\Lambda_1^{+'} \Lambda_2^{+'}}{P_{\parallel D} - \epsilon'_1 - \epsilon'_2} + \frac{\Lambda_1^{-'} \Lambda_2^{+'}}{-\epsilon'_1 - \epsilon_1 - \omega} \right) \frac{1}{P_{\parallel D} - \epsilon_1 - \epsilon'_2 - \omega} \left(\frac{\Lambda_1^+ \Lambda_2^+}{P_{\parallel D} - \epsilon_1 - \epsilon_2} + \frac{\Lambda_1^+ \Lambda_2^-}{-\epsilon_2 - \epsilon'_2 - \omega} \right) \right. \\
+ \left(\frac{\Lambda_1^{+'} \Lambda_2^{+'}}{P_{\parallel D} - \epsilon'_1 - \epsilon'_2} + \frac{\Lambda_1^{+'} \Lambda_2^{-'}}{-\epsilon_2 - \epsilon'_2 - \omega} \right) \frac{1}{P_{\parallel D} - \epsilon'_1 - \epsilon_2 - \omega} \left(\frac{\Lambda_1^+ \Lambda_2^+}{P_{\parallel D} - \epsilon_1 - \epsilon_2} + \frac{\Lambda_1^- \Lambda_2^+}{-\epsilon'_1 - \epsilon_1 - \omega} \right) \\
+ \frac{\Lambda_1^{-'} \Lambda_2^{-'}}{-P_{\parallel D} - \epsilon'_1 - \epsilon'_2} \left(\frac{1}{-\epsilon'_1 - \epsilon_1 - \omega} + \frac{1}{-\epsilon'_2 - \epsilon_2 - \omega} \right) \frac{\Lambda_1^+ \Lambda_2^+}{P_{\parallel D} - \epsilon_1 - \epsilon_2} \\
+ \left(\frac{\Lambda_1^{-'} \Lambda_2^{-'}}{-P_{\parallel D} - \epsilon'_1 - \epsilon'_2} + \frac{\Lambda_1^{+'} \Lambda_2^{-'}}{-\epsilon_1 - \epsilon'_1 - \omega} \right) \frac{1}{-P_{\parallel D} - \epsilon_1 - \epsilon'_2 - \omega} \left(\frac{\Lambda_1^- \Lambda_2^-}{-P_{\parallel D} - \epsilon_1 - \epsilon_2} + \frac{\Lambda_1^- \Lambda_2^+}{-\epsilon_2 - \epsilon'_2 - \omega} \right) \\
+ \left(\frac{\Lambda_1^{-'} \Lambda_2^{-'}}{-P_{\parallel D} - \epsilon'_1 - \epsilon'_2} + \frac{\Lambda_1^{-'} \Lambda_2^{+'}}{-\epsilon_2 - \epsilon'_2 - \omega} \right) \frac{1}{-P_{\parallel D} - \epsilon'_1 - \epsilon_2 - \omega} \left(\frac{\Lambda_1^- \Lambda_2^-}{-P_{\parallel D} - \epsilon_1 - \epsilon_2} + \frac{\Lambda_1^+ \Lambda_2^-}{-\epsilon_1 - \epsilon'_1 - \omega} \right) \\
\left. + \frac{\Lambda_1^{+'} \Lambda_2^{+'}}{P_{\parallel D} - \epsilon'_1 - \epsilon'_2} \left(\frac{1}{-\epsilon'_1 - \epsilon_1 - \omega} + \frac{1}{-\epsilon'_2 - \epsilon_2 - \omega} \right) \frac{\Lambda_1^- \Lambda_2^-}{-P_{\parallel D} - \epsilon_1 - \epsilon_2} \right], \quad (2.18)
\end{aligned}$$

where $\epsilon'_i \equiv \sqrt{m_i^2 - p_{i\perp D}^2}$, $\Lambda_i^{\pm'} = \Lambda_i^{\pm}(p'_{i\perp D})$, $\omega = \omega(p_{\perp D} - p'_{\perp D})$.

The interaction K_1 can be systematically improved by including the next terms in the coupling constant expansion of Eq. (2.11). In general, the K_1 defined by Eq. (2.11) is the two-particle irreducible interaction, with two-particle irreducibility defined with respect to the propagator $\langle G_0 \rangle$.

As in the Klein approach (see Refs. [26–29]) the three-dimensional Green's function $\langle G_0 + G_0 T G_0 \rangle$, where T is the Bethe-Salpeter amplitude, may be expanded as a series of diagrams. From the arguments given so far in this section, it is now clear that the Klein rules discussed in Refs. [25–27], will only be modified by the following kinematical replacements:

$$E \rightarrow P_{\parallel D}; \quad (2.19)$$

$$\mathbf{p}_i \rightarrow p_{i\perp D}; \quad (2.20)$$

$$\epsilon_i(\mathbf{p}_i) \rightarrow \epsilon_i(p_{i\perp D}) \equiv \sqrt{m_i^2 - p_{i\perp D}^2}; \quad (2.21)$$

$$\omega(\mathbf{p} - \mathbf{p}') \rightarrow \omega(p_{\perp D} - p'_{\perp D}) \equiv \sqrt{\mu^2 - (p_{\perp D} - p'_{\perp D})^2}. \quad (2.22)$$

Any choice for K_1 may therefore be expressed as a set of diagrams and calculated in a straightforward way. Once again, we observe that if the non-covariant choice (2.4) is made for D , then these replacements have no effect. Since the Klein rules coincide with those of time-ordered perturbation theory in $++$ states, an interaction generated using a particular set of diagrams in the formalism discussed here with this choice for D will agree with the time-ordered perturbation theory result for the same diagrams, provided only positive-energy states are considered.

However, unlike either the time-ordered perturbation theory or Klein interactions, K_1 may be defined so that it boosts in a covariant way. Consider, for instance, the choice $D = P$. The result of the above procedure is then a covariant three-dimensional interaction for use in Eq. (2.12). In the c.m. frame, where $\hat{P} = (1, 0, 0, 0)$ this interaction agrees with the Klein result. After solving for Γ_1 in the c.m. frame we can simply make the replacement $\mathbf{p} \rightarrow p_{\perp P}$ and thereby obtain a covariant vertex function in any frame. Once this equation is defined in one frame a kinematic boost defines it in all other frames.

If D is not chosen to be parallel to the total four-momentum of the two-body system then vertex functions in an arbitrary frame cannot be obtained from a kinematical boost of the rest-frame vertex function with the same D . To see this, consider some arbitrary choice for the four-vector D . Suppose that P is chosen to be a four-vector describing the deuteron at rest and that we wish to calculate $\Gamma_1(k_{\perp D}; P')$. Under a Lorentz transformation \mathcal{L}

$$\Gamma_1([\mathcal{L}k]_{D'}; \mathcal{L}P) = S(\mathcal{L})\Gamma_1(k_{\perp D}; P), \quad (2.23)$$

where $D' = \mathcal{L}D$, $S(\mathcal{L}) = S_1(\mathcal{L})S_2(\mathcal{L})$, and S_i is the unitary representation of the boost \mathcal{L} acting on the spin-half particle i . Consequently, $\Gamma_1(k_{\perp D}; P')$ can be related by a Lorentz boost to $\Gamma_1(k_{\perp D_0}; P)$, where \mathcal{L} is chosen such that $P' = \mathcal{L}P$, and $D_0 \equiv \mathcal{L}^{-1}D$. Therefore Γ can always be related to a rest-frame vertex function, but, unless D is chosen to be the total four-momentum of the two-body system, this rest-frame vertex function corresponds to a reduction of the four-dimensional Bethe-Salpeter vertex different from that used in calculating $\Gamma_1(k_{\perp D}; P)$.

When a truncated kernel K_1 is used this difference matters, since then for each choice of D the kernel of the bound-state equation (2.12) is different. Consequently, the bound-state equation must be re-solved at each value of P' in order to determine the vertex function $\Gamma(k_{\perp D}; P')$. If a truncation of the kernel K_1 is employed, the solutions for values of $P' \neq P$ will not correspond to the same bound-state mass as the vertex function $\Gamma_1(k_{\perp D}; P)$. It is possible to minimize this effect by using a sufficiently high-order truncation of the kernel. A simpler expedient is to use a low-order truncation and correct this by rescaling the kernel as a function of D so as to maintain the same invariant mass for the initial and final states of Eq. (3.14). Because of Eq. (2.23) this rescaling may equivalently be viewed as a function of the total four-momentum P' .

III. CONSTRUCTION OF A THREE-DIMENSIONAL GAUGE INVARIANT AMPLITUDE

In Section II we showed how to derive a three-dimensional two-body bound-state equation from the corresponding four-dimensional Bethe-Salpeter equation. With the choice $D = P$

this method maintains the Lorentz invariance of the original theory. The calculations of Refs. [25,33] indicate that this formalism holds promise for accurately approximating the results of Bethe-Salpeter equations. However, in any reduction of a gauge-invariant, four-dimensional theory to three dimensions, a critical test of the reduction procedure is whether it can consistently maintain the gauge invariance of the full four-dimensional theory. In this section we first consider the Green's function for the interaction of two particles with the photon constructed from the Bethe-Salpeter Green's function. The general construction of currents guarantees that this four-dimensional Green's function obeys a Ward-Takahashi identity (WTI), as required for gauge invariance. (Obtaining such a WTI in the four-dimensional theory in general requires the inclusion of meson-exchange currents.) We then turn our attention to the corresponding three-dimensional formalism, and show that truncating the two-body current and the two-body interaction K_1 at the same order in the coupling constant leads to a formalism for the calculation of electromagnetic processes which is both gauge invariant and Lorentz invariant.

We begin our discussion by reviewing the WTI of the Bethe-Salpeter Green's function. Consider the two-body Green's function G that is the solution of the Bethe-Salpeter equation

$$G = G_0 + G_0 K G. \quad (3.1)$$

Define the Green's function for the interaction of a free single particle i with a photon of fixed momentum Q via:

$$g_{0\mu}^{(i)}(p_i, Q) = d_i(p_i + Q) j_\mu^{(i)}(Q^2) d_i(p_i). \quad (3.2)$$

Here the operator $j_\mu^{(i)}$ is assumed to be such that this one-body photon Green's function obeys a WTI [34,35]

$$Q^\mu g_{0\mu}^{(i)}(p_i, Q) = e_i(d_i(p_i) - d_i(p_i + Q)). \quad (3.3)$$

Defining the two-body analog of Eq. (3.2):

$$\begin{aligned} G_0(p_1 + Q, p_2) j_\mu^{(1)}(Q^2) d_2^{-1}(p_2) G_0(p_1, p_2) + G_0(p_1, p_2 + Q) d_1^{-1}(p_1) j_\mu^{(2)}(Q^2) G_0(p_1, p_2) \\ \equiv G_0(p_1 + Q, p_2) j_\mu^{(1)}(Q^2) d_2^{-1}(p_2) G_0(p_1, p_2) + (1 \leftrightarrow 2) \equiv G_{0\mu}^\gamma(p_1, p_2, Q). \end{aligned} \quad (3.4)$$

Here, and throughout the rest of the paper, the notation $(1 \leftrightarrow 2)$ indicates that the momenta of the two particles must be swapped, *and* the labels exchanged. Therefore, the $(1 \leftrightarrow 2)$ pieces of any expression represent the photon coupling to whichever particle it did not couple to in the first part of the expression. We observe that the Green's function $G_{0\mu}^\gamma$ also obeys a WTI

$$Q^\mu G_{0\mu}^\gamma(p_1, p_2, Q) = e_1(G_0(p_1, p_2) - G_0(p_1 + Q, p_2)) + (1 \leftrightarrow 2). \quad (3.5)$$

Now let G_μ^γ be the Green's function for the interaction of one photon with the interacting two-particle system. Note that in the two-nucleon system the quantities e_i will include isospin operators, so care must be exercised in ordering charges and any interactions which also involve isospin operators. We may write the following equation for a gauge invariant G_μ^γ , by simply allowing the photon to be inserted anywhere on the right-hand side of Eq. (3.1),

$$G_\mu^\gamma = G_{0\mu}^\gamma + G_{0\mu}^\gamma K G + G_0 K G_\mu^\gamma + G_0 K_\mu^\gamma G. \quad (3.6)$$

Here K_μ^γ is found by coupling the photon to every internal charged line in the kernel K .

Using Eq. (3.1) and the definition (3.4), Eq. (3.6) may easily be solved for G_μ^γ

$$\begin{aligned} G_\mu^\gamma(k'_1, k'_2; k_1, k_2; Q) &= \int \frac{d^4 p_1 d^4 p_2}{(2\pi)^8} G(k'_1, k'_2; p_1 + Q, p_2) j_\mu^{(1)}(Q^2) d_2^{-1}(p_2) G(p_1, p_2; k_1, k_2) \\ &+ (1 \leftrightarrow 2) + \int \frac{d^4 p'_1 d^4 p'_2 d^4 p_1 d^4 p_2}{(2\pi)^{16}} G(k'_1, k'_2; p'_1, p'_2) K_\mu^\gamma(p'_1, p'_2; p_1, p_2; Q) G(p_1, p_2; k_1, k_2). \end{aligned} \quad (3.7)$$

(Note that there is an overall delta function $\delta^{(4)}(k'_1 + k'_2 - k_1 - k_2 - Q)$ on both sides of this equation.) Using the identity (3.5) in Eq. (3.7) and the explicit form of Eq. (3.1) we find

$$\begin{aligned} Q^\mu G_\mu^\gamma(k'_1, k'_2; k_1, k_2; Q) &= e_1 G(k'_1 - Q, k'_2; k_1, k_2) - G(k'_1, k'_2; k_1 + Q, k_2) e_1 \\ &+ \int \frac{d^4 p'_1 d^4 p'_2 d^4 p_1 d^4 p_2}{(2\pi)^{16}} G(k'_1, k'_2; p'_1, p'_2) [K(p'_1, p'_2; p_1 + Q, p_2) e_1 \\ &\quad - e_1 K(p'_1 - Q, p'_2; p_1, p_2)] G(p_1, p_2; k_1, k_2) + (1 \leftrightarrow 2) \\ &+ \int \frac{d^4 p'_1 d^4 p'_2 d^4 p_1 d^4 p_2}{(2\pi)^{16}} G(k'_1, k'_2; p'_1, p'_2) Q^\mu K_\mu^\gamma(p'_1, p'_2; p_1, p_2; Q) G(p_1, p_2; k_1, k_2). \end{aligned} \quad (3.8)$$

This reduces to a WTI for G_μ^γ ,

$$Q^\mu G_\mu^\gamma(k'_1, k'_2; k_1, k_2; Q) = e_1 G(k'_1 - Q, k'_2; k_1, k_2) - G(k'_1, k'_2; k_1 + Q, k_2) e_1 + (1 \leftrightarrow 2), \quad (3.9)$$

provided that

$$e_1 K(p'_1 - Q, p'_2; p_1, p_2) - K(p'_1, p'_2; p_1 + Q, p_2) e_1 + (1 \leftrightarrow 2) = Q^\mu K_\mu^\gamma(p'_1, p'_2; p_1, p_2; Q), \quad (3.10)$$

which is the WTI for the interaction current. (Similar identities are used in the construction of a gauge invariant electromagnetic matrix element for the Gross—or spectator—formalism in Refs. [36,37].) The result (3.10) is completely general, and will always hold if the two-body current K_μ^γ is constructed in a gauge invariant way. In Appendix A we show how Eq. (3.10) is achieved in the special case where the photon couples only to particle one, i.e. the exchanged particles and particle two carry no charge. By contrast, if the particles are, for instance, nucleons carrying isospin which exchange isovector mesons, the analysis is more complicated, partly because the isospin operators in K and G do not commute with those in the charges e_1 and e_2 . However, as alluded to above, the additional terms generated because of this non-commutativity are canceled by terms arising from the coupling of the photon to isovector mesons. (The interested reader may consult Ref. [36] for details on this problem.)

Now, in the case of a bound state we have the following decomposition of the Green's function

$$G(k'_1, k'_2; k_1, k_2) = \delta^{(4)}(k'_1 + k'_2 - k_1 - k_2) \left[G_0(k'_1, k'_2) \frac{\Gamma(k'_1, k'_2) \bar{\Gamma}(k_1, k_2)}{P^2 - M^2} G_0(k_1, k_2) + R(k'_1, k'_2; k_1, k_2) \right], \quad (3.11)$$

where $P = k_1 + k_2 = k'_1 + k'_2$ is the conserved total four-momentum of the two-body system, M is the two-body bound-state mass, and the piece R is regular at $P^2 = M^2$. We then insert Eq. (3.11) into (3.7) and extract the double-pole part of the resulting expression, in order to obtain the amplitude for interaction of the bound-state with a photon of momentum Q [38]. Expressing the result in terms of total and relative four-momenta yields:

$$\begin{aligned} \mathcal{A}_\mu(P, Q) = & \int \frac{d^4 p}{(2\pi)^4} \bar{\Gamma}(p + Q/2; P + Q) G_{0\mu}^\gamma(p; P, Q) \Gamma(p; P). \\ & + \int \frac{d^4 p' d^4 p}{(2\pi)^8} \bar{\Gamma}(p'; P + Q) G_0(p'; P + Q) K_\mu^\gamma(p', P + Q; p, P; Q) G_0(p; P) \Gamma(p; P). \end{aligned} \quad (3.12)$$

From Eq. (3.5), Eq. (3.10), and the bound-state BSE, Eq. (2.1), it is straightforward to show that

$$Q^\mu \mathcal{A}_\mu(P, Q) = 0, \quad (3.13)$$

as required by gauge invariance.

In the context of the formalism of Section II the question is how to maintain this gauge invariance when the reduction to three dimensions is made. In particular, in Section II we showed how Γ_1 , which obeys a three-dimensional integral equation, could be used to find Γ . Inserting the expression (2.10) into Eq. (3.12) and using the definition of \mathcal{G} we find

$$\begin{aligned} \mathcal{A}_\mu(P, Q) = & \int \frac{d^4 k' d^4 p' d^4 p d^4 k}{(2\pi)^{16}} \bar{\Gamma}_1(k'_{\perp D}; P + Q) \mathcal{G}(k', p'; P + Q) \\ & \left[j_\mu^{(1)}(Q^2) d_2^{-1}(\nu_2 P - p) \delta^{(4)}(p' - p - Q/2) + j_\mu^{(2)}(Q^2) d_1^{-1}(\nu_1 P - p) \delta^{(4)}(p' - p + Q/2) \right. \\ & \left. + K_\mu^\gamma(P + Q, p'; P, p; Q) \right] \mathcal{G}(p, k; P) \Gamma_1(k_{\perp D}; P). \end{aligned} \quad (3.14)$$

An important part of Eq. (3.14) is that we use the same D in the initial and final states. From the discussion of Section II, it might appear that the natural choices are $D = P$ in the initial state, and $D = P + Q$ in the final state. However, if these choices are made then the formula (3.14) is actually slightly misleading, because the two vertex functions so defined depend on pieces of the four-vectors k and k' which reside in different subspaces. Thus, for instance, the two \mathcal{G} 's appearing in Eq. (3.14) are actually different objects. This immediately leads to significant complications. In particular it makes it very difficult to define a three-dimensional gauge invariant approximation to the amplitude (3.12).

In this current matrix element, the choice

$$D = (P + P')/2, \quad (3.15)$$

may be made in both the initial and final state. With this choice the vertex functions $\Gamma_1(k_{\perp D}, P + Q)$ and $\Gamma_1(k_{\perp D}, P)$ cannot both be obtained from a kinematical boost of the rest frame wave functions with $D = (1, 0, 0, 0)$. However, in spite of the resultant variation in the bound-state mass discussed in Section II, the matrix element (3.14) is covariant provided that *both* the vectors P and P' are boosted using the same Lorentz transformation \mathcal{L} . This is easily shown by expanding Eq. (3.14) in powers of $K - K_1$ and then using Eq. (2.23), the definition of K_1 , and the known covariance properties of K and G_0 .

The choice (3.15) is a useful one because it allows the straightforward definition of a three-dimensional matrix element for the interaction of the bound state with an electromagnetic probe which is not only covariant, but is also gauge invariant. To show this, first note that in order to solve the integral equation (2.8), we in fact choose K_1 so as to impose

$$\langle \mathcal{G} \rangle = \langle G_0 \rangle. \quad (3.16)$$

If this condition is imposed to all orders in the coupling constant then the result is an infinite series for K_1 . If we impose the condition (3.16) order-by-order in the expansion in $K - K_1$, the condition defines K_1 to the same order. Truncation of the kernel is necessary for a practical analysis and it becomes crucial to inquire if a corresponding approximation for the matrix element (3.14) exists that maintains the gauge invariance of the theory. It turns out that the current matrix element (3.14) is gauge invariant if $\mathcal{G}(J_\mu + K_\mu^\gamma)\mathcal{G}$ on the right-hand side of Eq. (3.14) is expanded to a given order in the coupling constant and the kernel K_1 used to define Γ_1 is obtained from Eq. (3.16) by truncation at the same order in the coupling constant.

To see this we first abbreviate Eq. (3.14) as:

$$\mathcal{A}_\mu(P, Q) = \bar{\Gamma}_1(P + Q)\mathcal{G}(P + Q)(J_\mu + K_\mu^\gamma)\mathcal{G}(P)\Gamma_1(P), \quad (3.17)$$

where all integrations over relative momenta are now suppressed. We split this into two pieces, one due to the one-body current J_μ , and one due to the two-body current K_μ^γ . Suppose that K_1 has been truncated at lowest order, i.e. $K_1 = K_1^{(2)}$. Then, in the J_μ piece, we expand the \mathcal{G} s and retain terms up to first order in $K - K_1$. To this we add a piece from the two-body current, in which we stop this expansion of \mathcal{G} at zeroth order in $K - K_1$, i.e., write $\mathcal{G} = G_0$. Thus, we define a first-order approximation to \mathcal{A}_μ , $\mathcal{A}_\mu^{(1)}$, by

$$\begin{aligned} \mathcal{A}_\mu^{(1)} = & \bar{\Gamma}_1(P + Q)\langle G_{0\mu}^\gamma \rangle \Gamma_1(P) + \bar{\Gamma}_1(P + Q)\langle G_0(P + Q)(K(P + Q) - K_1(P + Q))G_{0\mu}^\gamma \rangle \Gamma_1(P) \\ & + \bar{\Gamma}_1(P + Q)\langle G_{0\mu}^\gamma(K(P) - K_1(P))G_0(P) \rangle \Gamma_1(P) \\ & + \bar{\Gamma}_1(P + Q)\langle G_0(P + Q)K_\mu^\gamma G_0(P) \rangle \Gamma_1(P). \end{aligned} \quad (3.18)$$

It follows from Eq. (2.23) that $\mathcal{A}_\mu^{(1)}$ is a covariant matrix element. To show $\mathcal{A}_\mu^{(1)}$ is also gauge invariant we contract with the four-vector Q and use the identities (3.5) and (3.10). This leads to

$$\begin{aligned} Q^\mu \mathcal{A}_\mu^{(1)} = & \bar{\Gamma}_1(P + Q)e_1\langle G_0(P) \rangle \Gamma_1(P) - \bar{\Gamma}_1(P + Q)\langle G_0(P + Q) \rangle e_1\Gamma_1(P) \\ & + \bar{\Gamma}_1(P + Q)\langle G_0(P + Q)(K(P + Q) - K_1(P + Q))e_1G_0(P) \rangle \Gamma_1(P) \\ & - \bar{\Gamma}_1(P + Q)\langle G_0(P + Q)(K(P + Q) - K_1(P + Q))G_0(P + Q) \rangle e_1\Gamma_1(P) \\ & + \bar{\Gamma}_1(P + Q)e_1\langle G_0(P)(K(P) - K_1(P))G_0(P) \rangle \Gamma_1(P) \\ & - \bar{\Gamma}_1(P + Q)\langle G_0(P + Q)e_1(K(P) - K_1(P))G_0(P) \rangle \Gamma_1(P) \\ & + \bar{\Gamma}_1(P + Q)\langle G_0(P + Q)(e_1K(P) - K(P + Q)e_1)G_0(P) \rangle \Gamma_1(P) + (1 \leftrightarrow 2). \end{aligned} \quad (3.19)$$

The bound-state equation (2.12) can now be used to show that the terms in the first two lines cancel the terms in the last two lines, and so:

$$\begin{aligned} Q^\mu \mathcal{A}_\mu^{(1)} = & -\bar{\Gamma}_1(P + Q)\langle G_0(P + Q)(K(P + Q) - K_1(P + Q))G_0(P + Q) \rangle e_1\Gamma_1(P) \\ & + \bar{\Gamma}_1(P + Q)e_1\langle G_0(P)(K(P) - K_1(P))G_0(P) \rangle \Gamma_1(P) + (1 \leftrightarrow 2). \end{aligned} \quad (3.20)$$

But, at second-order in the coupling constant, Eq. (3.16) leads to Eq. (2.15) for K_1 . It immediately follows that if K_1 has been defined in this way, the corresponding amplitude for electromagnetic interactions of the bound state, as defined by Eq. (3.18), obeys

$$Q^\mu \mathcal{A}_\mu^{(1)} = 0. \quad (3.21)$$

It is straightforward to check that the same result holds if Eq. (2.12) is truncated at next order, while the one-body current pieces of Eq. (3.14) are expanded to second order in $K - K_1$ and the two-body current pieces are expanded to first order in $K - K_1$. That is, if the interaction $K_1^{(4)}$ defined by (2.16) is used, then defining a vertex function $A_\mu^{(2)}$ by expanding Eq. (3.17) to fourth order in the coupling constant leads to $Q^\mu \mathcal{A}_\mu^{(2)} = 0$ also. Thus, we conclude that truncating the kernel K_1 defined by Eq. (3.16) and the electromagnetic vertex defined by Eq. (3.17) at consistent order in the coupling constant yields a gauge invariant, covariant, electromagnetic matrix element.

It should be pointed out that $\mathcal{A}_\mu^{(1)}$ includes contributions from diagrams where the photon couples to particles one and two while exchanged quanta are “in-flight”. These contributions are of two kinds. Firstly, if the four-dimensional kernel K is dependent on the total momentum then gauge invariance requires the presence of terms representing the coupling of the photon to internal lines in K . Secondly, even if the kernel does not depend on the total four-momentum, e.g. it is a OBE kernel, terms arise in the three-dimensional formalism where the photon couples to particles one and two while an exchanged particle is “in-flight”. These must be included if our approach is to contain a WTI. (See Fig. 1 for a diagrammatic interpretation of one such term.) Many years ago, in the course of work on deuteron photo-disintegration, Pearlstein and Klein [39] showed that contributions such as these had to be included if Siegert’s theorem (which is a consequence of gauge invariance) was to hold in the Klein approach. Our systematic derivation shows the general form of the additional pieces of the current operator which are required if gauge invariance is to be maintained.

Gross and Riska [36] and Kvinikhidze and Blankleider [37] have discussed the construction of gauge invariant amplitudes for the interaction of a photon with the two-body system within the Gross (or spectator) quasipotential approach [40,41]. It would be interesting to compare the systematic approach to constructing electromagnetic matrix elements developed here with the discussions of Refs. [36,37].

IV. CONCLUSION

In this paper, a systematic formalism is presented for the reduction of the Bethe-Salpeter equation to a three-dimensional form in which components of the relative momentum parallel to a direction D are integrated out. By choosing $D = P$, where P is the total four-momentum of the two-body system, a covariant formalism is obtained for a bound state.

This part of the development follows closely the analysis of Ref. [25], where the corresponding formalism based on the choice $D = (1, 0, 0, 0)$ was constructed. However, here we simplify the analysis by omitting the treatment of crossed graphs in order to focus on covariance and gauge invariance.

Consideration of the matrix elements of a conserved current shows that the gauge invariance of the underlying four-dimensional theory is preserved in the three-dimensional

reduction. Moreover, it is possible to truncate both the infinite series expansion for the interaction kernel used to define the bound states and the expansion of the currents at the same order in the coupling constant, and maintain this gauge invariance.

The three-dimensional reduction for current matrix elements is based upon the use of the same direction D in initial and final states. Here, the choice $D = \frac{1}{2}(P + P')$ is suggested, where P (P') is the initial (final) total four-momentum of the two-body system. Other possible choices, e.g. $D = P$ or $D = P'$, could be used just as well, and may be preferable in some circumstances. The current matrix elements are covariant for any choice of D that is a linear combination of the two independent four-vectors P and P' .

When truncations of the interaction kernel are employed in order to implement a practical calculation the results depend on the choice of D . The dependence on D in a truncated kernel, K_1 , can be reduced systematically by keeping higher-order terms. Alternatively, the invariance of the mass of initial and final bound states can be enforced by a rescaling of the coupling constants in K_1 .

There are two important advantages of the reduction formalism based on integrating out momentum components presented here. Firstly, it is systematic. Secondly, as discussed in Ref. [25], it avoids unphysical singularities. Such singularities generally arise in systematic quasipotential reductions to three dimensions. Typically such reductions involve splitting the propagator into a part involving a delta function and a remainder. Corrections to the theory have unphysical singularities, such as particle production thresholds at energies below the physical values. Moreover, current matrix elements obtained using quasipotential reductions generally involve similar unphysical singularities which must be either disregarded or suppressed, with uncertain consequences. Such difficulties have motivated the present approach. This systematic formalism in which gauge invariance can be maintained by consistent approximations in the current operator and interaction kernel should prove to be a useful tool in the calculation of relativistic bound state properties.

ACKNOWLEDGMENTS

We are grateful to the U. S. Department of Energy for its support under grant no. DE-FG02-93ER-40762. We also both thank Professor Tjon for a number of useful and stimulating conversations on this topic, and on other topics in relativistic few-body physics. D. R. P. thanks Iraj Afnan for useful correspondence on this paper.

APPENDIX A: PROOF OF WARD-TAKAHASHI IDENTITY FOR K_μ^γ IN A SPECIAL CASE

In what follows we assume that the photon couples only to particle one. If the interaction between particles one and two is mediated by quanta which carry charge, then the discussion given here must be modified to incorporate terms where the photons couple to the exchanged quanta. For instance, if particles one and two are nucleons carrying isospin which exchange isovector mesons, the analysis is more complicated, due to the non-commutativity of isospin operators in K and G with those in the charges e_1 and e_2 . However, the additional terms

generated via this complication are canceled by terms arising from the direct coupling of the photon to the isovector mesons [36].

Consider the n th order piece of the kernel K . If $e_2 = 0$ and the exchanged particles do not couple to the photon we need only concern ourselves with the particle one pieces of the three terms in Eq. (3.10), since the particle two and meson propagators may be arranged so as to take exactly the same form in all three pieces. Suppose then that the momenta on internal lines of particle one are labeled $p_1^{(1)}, p_1^{(2)}, \dots, p_1^{(j)}$ to the right of the photon insertion, and $p_1^{(j)} + Q, p_1^{(j+1)} + Q, \dots, p_1^{(n-1)} + Q$ to the left of the photon insertion. Meanwhile, the vertices governing the interaction of the exchanged quanta with particle one are labeled $V_1^{(1)}, \dots, V_1^{(n)}$. With these conventions we may write the particle one pieces of the three terms in Eq. (3.10) as:

$$K(p'_1, p'_2; p_1 + Q, p_2) : V_1^{(n)} \prod_{i=1}^{n-1} d_1(p_1^{(i)} + Q) V_1^{(i)} \quad (\text{A1})$$

$$K(p'_1 - Q, p'_2; p_1, p_2) : V_1^{(n)} \prod_{i=1}^{n-1} d_1(p_1^{(i)}) V_1^{(i)} \quad (\text{A2})$$

$$K_\mu^\gamma(p'_1, p'_2; p_1, p_2; Q) : \sum_{j=1}^{n-1} V_1^{(n)} \left[\prod_{i=j+1}^{n-1} d_1(p_1^{(i)} + Q) V_1^{(i)} \right] g_{0\mu}^{(1)}(p_1^{(j)}, Q) V_1^{(j)} \left[\prod_{i=1}^{j-1} d_1(p_1^{(i)}) V_1^{(i)} \right], \quad (\text{A3})$$

where the product symbols are defined to give one if the upper index of the product lies below the lower index. Note that the changes in particle one's external momenta guarantee that the particle two and meson pieces of the diagram will be identical in all three cases.

Now calculating $Q^\mu K_\mu^\gamma$ using the identity (3.3) we obtain, for the particle one pieces of this object:

$$\begin{aligned} & \sum_{j=1}^{n-1} V_1^{(n)} \left[\prod_{i=j+1}^{n-1} d_1(p_1^{(i)} + Q) V_1^{(i)} \right] e_1 d_1(p_1^{(j)}) V_1^{(j)} \left[\prod_{i=1}^{j-1} d_1(p_1^{(i)}) V_1^{(i)} \right] \\ & - \sum_{j=1}^{n-1} V_1^{(n)} \left[\prod_{i=j+1}^{n-1} d_1(p_1^{(i)} + Q) V_1^{(i)} \right] d_1(p_1^{(j)} + Q) e_1 V_1^{(j)} \left[\prod_{i=1}^j d_1(p_1^{(i)}) V_1^{(i)} \right]. \end{aligned} \quad (\text{A4})$$

This is a telescoping series, as the j th term from the first piece cancels the $j + 1$ th term from the second piece. The only structures which survive this cancelation give precisely the particle one piece of

$$e_1 K(p'_1 - Q, p'_2; p_1, p_2) - K(p'_1, p'_2; p_1 + Q, p_2) e_1. \quad (\text{A5})$$

Note that the series can only be summed in this way if the vertices commute with the charge operator e_1 . However, if the particles are distinguishable and $[V_1^{(j)}, e_1] = 0$ then this argument suffices to prove Eq. (3.10).

REFERENCES

- [1] J. Fleischer and J. A. Tjon, Nucl. Phys. **B84**, 375 (1975).
- [2] J. Fleischer and J. A. Tjon, Phys. Rev. D **15**, 2537 (1977).
- [3] J. Fleischer and J. A. Tjon, Phys. Rev. D **21**, 87 (1980).
- [4] M. J. Zuilhof and J. A. Tjon, Phys. Rev. C **22**, 2369 (1980).
- [5] M. J. Zuilhof and J. A. Tjon, Phys. Rev. C **24**, 736 (1981).
- [6] J. A. Tjon and E. van Faassen, Phys. Lett. B. **120**, 39 (1983).
- [7] E. van Faassen and J. A. Tjon, Phys. Rev. C **30**, 285 (1984).
- [8] G. Rupp and J. A. Tjon, Phys. Rev. C **37**, 1729 (1988).
- [9] G. Rupp and J. A. Tjon, Phys. Rev. C **45**, 2133 (1992).
- [10] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
- [11] M. Gell-Mann and F. E. Low, Phys. Rev. **84**, 350 (1951).
- [12] Y. Nambu, Prog. Theor. Phys. **5**, 614 (1950).
- [13] J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452 (1951).
- [14] J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 455 (1951).
- [15] E. E. Salpeter, Phys. Rev. **87**, 328 (1952).
- [16] R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966).
- [17] A. A. Logunov and A. N. Tavkhelidze, Nuovo Cim. **29**, 380 (1963).
- [18] E. Hummel and J. A. Tjon, Phys. Rev. Lett. **63**, 1788 (1989).
- [19] E. Hummel and J. A. Tjon, Phys. Rev. C **42**, 423 (1990).
- [20] E. Hummel and J. A. Tjon, Phys. Rev. C **49**, 21 (1994).
- [21] G. H. Martinus, O. Scholten, and J. A. Tjon, Phys. Lett. B **402**, 7 (1997).
- [22] P. Tiemeijer and J. A. Tjon, Phys. Lett. B **277**, 38 (1992).
- [23] P. Tiemeijer and J. A. Tjon, Phys. Rev. C **48**, 896 (1993).
- [24] P. Tiemeijer and J. A. Tjon, Phys. Rev. C **49**, 494 (1994).
- [25] D. R. Phillips and S. J. Wallace, Phys. Rev. C **54**, 507 (1996).
- [26] A. Klein, Phys. Rev. **90**, 1101 (1953).
- [27] A. Klein, Phys. Rev. **94**, 1052 (1954).
- [28] A. Klein and B. H. McCormick, Prog. Theor. Phys. **20**, 876 (1958).
- [29] A. Klein and T.-S. H. Lee, Phys. Rev. D **10**, 4308 (1974).
- [30] V. B. Mandelzweig and S. J. Wallace, Phys. Lett. B. **197**, 469 (1987).
- [31] S. J. Wallace, in *Nuclear and Particle Physics on the Light Cone*, edited by M. B. Johnson and L. S. Kisslinger (World Scientific, Singapore, 1988).
- [32] S. J. Wallace and V. B. Mandelzweig, Nucl. Phys. **A503**, 673 (1989).
- [33] A. D. Lahiff and I. R. Afnan (unpublished).
- [34] J. C. Ward, Phys. Rev. **78**, 182 (1950).
- [35] Y. Takahashi, Nuovo Cim. **6**, 371 (1957).
- [36] F. Gross and D. O. Riska, Phys. Rev. C **36**, 1928 (1987).
- [37] A. N. Kvinikhidze and B. Blankleider, nucl-th/9706051 (unpublished).
- [38] S. Mandelstam, Proc. Roy. Soc. (London) **233**, 248 (1955).
- [39] L. D. Pearlstein and A. Klein, Phys. Rev. **118**, 193 (1960).
- [40] F. Gross, Phys. Rev. C **26**, 2203 (1982).
- [41] J. W. van Orden, Czech. J. Phys. **45**, 181 (1995).

FIGURES

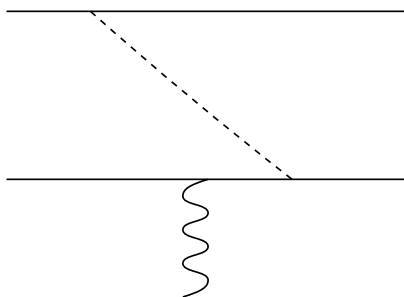


FIG. 1. One example of a graph which is required in our formalism in order to maintain gauge invariance.